The present note deals admittedly with questions whose satisfactory treatment would require a large amount of space. It is hoped, however, that a few points have become perfectly clear and that these may throw light on others. In the first place, there seems to be no room for doubt in regard to the inaccuracy of the figures appearing in various well and favorably known histories of mathematics which aim to present Gerbert's explanation of the formula $\frac{1}{2}a(a+1)$ for the area of an equilateral triangle whose side is a. A correct figure appears in Bubnov's *Gerberti Opera Mathematica*, 1899, Tab. I. In the second place, it should be clear that Gerbert's attempted explanation of the formula in question exhibits too little mathematical insight and is too trivial to merit the epithet which has been so widely attributed to it. While the time from the beginning of the Middle Ages to the end of the tenth century does not exhibit much creditable work in mathematics it does present some noteworthy advances, especially in algebra and trigonometry.

NOTE ON THE PRIME DIVISORS OF THE NUMERATORS OF BERNOULLI'S NUMBERS.

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1. Using the even-suffix notation for Bernoulli's numbers, $B_0 = 1$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $\cdots B_{12} = -691/2730$, $B_{14} = 7/6$, $B_{16} = -3617/510$, $\cdots$, as in Lucas, *Théorie des Nombres*, Chap. XIV, we shall prove the following

THEOREM. *If p is an odd prime which does not divide $4^r - 1$, the numerator of B_{2pr} is divisible by p.*

Hence for $r = 1$ we have a result due to John Couch Adams:[1]

COROLLARY. *If $p > 3$ is a prime, the numerator of B_{2p} is divisible by p.*

Both of these are useful as checks in numerical work, also they have a certain theoretical interest in some parts of arithmetic. Another observation due to Adams (quoted by Lucas, p. 435), states[2] that if p is an odd prime divisor of q and not a divisor of the denominator of B_{2q}, the numerator of B_{2q} is divisible by p. A comparison of this result and that which we shall establish shows that in numerical work one can often be applied with less labor than the other.

2. The proof depends upon the known fact that for $q > 0$ an integer,

$$I_q \equiv 2^{2q-1}(2^{2q} - 1)B_{2q}/q$$

is an integer.[3] Assume this for a moment; write $q=pr$, where p is an odd prime, and put $B_{2q} = N_{2q}/D_{2q}$, N_{2q}, D_{2q} being the numerator and denominator respectively of B_{2q}. Then

$$I_{pr} = 2^{2pr-1}(2^{2pr} - 1)N_{2pr}/prD_{2pr}.$$

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[1] *Scientific Papers of John Couch Adams*, vol. 1, 1896, pp. xliv, 430.—EDITOR.

[2] J. C. Adams, Crelle's *Journal*, vol. 85, 1878, p. 269; *Scientific Papers*, vol. 1, p. 430.—EDITOR.

[3] Compare *Encyklopädie der Mathematischen Wissenschaften*, vol. II–1, 2–3, 1899, p. 183.—EDITOR.

By Fermat's theorem

$$(2^{2r})^p - 2^{2r} = M(p), \text{ (a multiple of } p),$$

and hence

$$(2^{2r})^p - 1 = M(p) + (2^{2r} - 1),$$

the right-hand member of which, and therefore also the left, is a multiple of p when and only when $2^{2r} - 1$ is divisible by p. Obviously p cannot be a divisor of 2^{2pr-1} (excluding the trivial case $p = 1$); and hence since I_{pr} is an integer it follows that if p is not a divisor of $2^{2r} - 1$, then p must divide N_{2pr}; which is the theorem.

3. It doubtless is easy in many ways to show that I_q is an integer. We give the following for its suggestiveness: the simple remark that the coefficients in the k-polynomials are integers, when combined with less obvious properties of the elliptic integrals than that which is used here, leads to a rich and unexplored field for the Bernoulli and Euler numbers. This is particularly the case when the symbolic calculus of Blissard[1] (and Lucas) is applied to the formulas furnished by the theory of transformation.

Indicating in the usual manner the modulus of the elliptic function $\operatorname{sn} x$ by k and writing $\operatorname{sn}(x, k)$, we have for the modulus unity $\operatorname{sn}(x, 1)$, and it is easy to show (cf. Cayley, *Elliptic Functions*, p. 59) that $\operatorname{sn}(x, 1) = -i \tan ix$, where $i = \sqrt{-1}$, and therefore

$$2ix \operatorname{sn}(ix, 1) = -2x \tan x.$$

But, as may readily be seen on expanding by Maclaurin's theorem, the coefficient of $(-1)^n x^{2n+1}/(2n+1)!$ $(n \geqq 0)$ in the development of $\operatorname{sn}(x, k)$ is of the form

$$s_0 + s_1 k^2 + s_2 k^4 + \cdots + s_n k^{2n},$$

in which $s_0, s_1, \cdots, s_n$ are positive integers, and hence their sum S is a positive integer.

On the other hand it is well-known (cf. Lucas, *loc. cit.* p. 262) that the coefficient of $(-1)^n x^{2n}/(2n)!$ $(n \geqq 0)$ in the development of $2x \tan x$ is $2^{2n} G_{2n}$, where G_{2n} is the $2n$th Genocchi number[2] defined by

$$G_{2n} = 2(1 - 2^{2n}) B_{2n}.$$

Hence, equating coefficients of like powers of x in the two developments, we find

$$S = \frac{(-1)^{n+1} 4^n}{n+1} G_{2n+2} = \frac{(-1)^n 2^{2n+1}(2^{2n+2} - 1)}{n+1} B_{2n+2};$$

and therefore on writing $q = n + 1$, we have, in the notation of § 2, $I_q =$ the integer $(-1)^n S$.

[1] J. Blissard, *Quarterly Journal of Mathematics*, vols. 6–9, 1863–1867.—Editor.

[2] A. Genocchi, *Annali di Scienze Mathematische e Fisiche*, vol. 3, 1852, p. 395.—Editor.